

A GENERAL RECIPROCITY LAW ON ARBITRARY VECTOR SPACES

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ABSTRACT. The aim of this work is to offer a general theory of reciprocity laws on arbitrary vector spaces from commutators of central extensions of groups, and to show that classical explicit reciprocity laws are particular cases of this theory. Moreover, as another application of the general expression we provide a reciprocity law for characters.

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1. INTRODUCTION

In 1968 J. Tate [9] gave a definition of the residues of differentials on curves in terms of traces of certain linear operators on infinite-dimensional vector spaces. Furthermore, he proved the residue theorem (the additive reciprocity law) from the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$.

A few years later, in 1971, J. Milnor [3] defined the tame symbol $(\cdot, \cdot)_v$ associated with a discrete valuation v on a field F . Explicitly, if A_v is the valuation ring, p_v is the unique maximal ideal and $k_v = A_v/p_v$ is the residue class field, Milnor defined $(\cdot, \cdot)_v: F^\times \times F^\times \rightarrow k_v^\times$ by

$$(f, g)_v = (-1)^{v(f) \cdot v(g)} \frac{f^{v(g)}}{g^{v(f)}} \pmod{p_v}.$$

(Here and below R^\times denotes the multiplicative group of a ring R with unit.)

This definition generalizes the definition of the multiplicative local symbol given by J. P. Serre in [8]. If C is a complete, irreducible and non-singular curve over an algebraically closed field k , and Σ_C is its field of functions, the expression

$$\prod_{x \in C} (-1)^{v_x(f) \cdot v_x(g)} \frac{f^{v_x(g)}}{g^{v_x(f)}}(x) = 1$$

for all $f, g \in \Sigma_C^\times$, is the Weil Reciprocity Law.

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In 1989, Arbarello, De Concini and V.G. Kac [2]- provided a new definition of the tame symbol of an algebraic curve from the commutator of a certain central extension of groups and, analogously to Tate's construction, they deduce the Weil Reciprocity Law from the finiteness of its cohomology. More recently, in [6] the author has obtained a generalization of this result for the case of a complete curve over a perfect field. In both cases, similar to Tate's proof of the Residue Theorem, the reciprocity laws were proved by using "ad hoc" arguments from the properties of complete algebraic curves.

The goal of the present work is to offer a reciprocity law for commutators of families of k -subspaces of an arbitrary infinite-dimensional vector space, such that the above reciprocity laws for the tame symbol of a complete curve will be particular cases of this general theory. Moreover, we provide explicit examples of the given reciprocity law.

The organization of the paper is as follows. Section 2 contains a brief summary of the results on the commutators appeared in [1], [2] or [6], and new technical statements that will be that necessary tools for developing the theory of reciprocity laws on vector spaces offered. Section 3 is devoted to the main results of this work: proof of the general reciprocity law (Theorem 3.5); the definition of reciprocity-admissible families of vector subspaces that satisfy the conditions of the general reciprocity law, and, as an application, well-known explicit reciprocity laws that can be deduced from the general expression, including the Weil Reciprocity Law and the Reciprocity Law of the Hilbert Norm Residue Symbol for a complete curve over a finite field. Moreover, as another application of the general expression, we provide a reciprocity law for characters that allows us to recover the well-known formula $\sum_{x \in X} v_x(f) = 0$ for every $f \in \Sigma_X^\times$ with X a complete complex curve.

It should be noted that the statements in sections 2 and 3 are valid for each vector space over an arbitrary field and, therefore, it should be possible to study other reciprocity laws of symbols or to deduce new explicit expressions in Algebraic Number Theory using this method.

2. PRELIMINARIES

The first part of this section is added for the sake of completeness.

2.1. Commensurability on k -vector spaces. Let V be a vector space over a field k (in general, infinite-dimensional).

Definition 2.1. [9] *Two vector subspaces A and B of V are said to be commensurable if $\dim_k(A + B/A \cap B) < \infty$. We shall use the symbol $A \sim B$ to denote commensurable vector subspaces.*

Fixing a vector subspace, $V_+ \subset V$, if k is algebraically closed and

$$\mathrm{Gl}(V, V_+) = \{f \in \mathrm{Aut}_k(V) \text{ such that } f(V_+) \sim V_+\},$$

in 1989, E. Arbarello, C. de Concini and V.-G. Kac [2] constructed a determinantal central extension of groups:

$$(2.1) \quad 1 \rightarrow k^\times \longrightarrow \widetilde{\mathrm{Gl}(V, V_+)} \xrightarrow{\pi} \mathrm{Gl}(V, V_+) \rightarrow 1,$$

and they used this extension to study the tame symbol on an algebraic curve.

In 2002, the author of this work generalized the results of [2] to the case of vector spaces over an arbitrary ground field [6]-, and in 2004 G. W. Anderson and the

author offered in [1] the corresponding generalization to free A -modules, with A an Artinian local ring, using theory of groupoids and commensurability of A -modules.

Given an element $f \in \text{Gl}(V, V_+)$, the “index of f over V_+ ” is the integer number:

$$i(f, V_+, V) = \dim_k(V_+/V_+ \cap fV_+) - \dim_k(fV_+/V_+ \cap fV_+).$$

We denote by $\{\cdot, \cdot\}_{V_+}^V$ the commutator of the central extension (2.1); that is, if τ and σ are two commuting elements of $\text{Gl}(V, V_+)$ and $\widetilde{\tau}, \widetilde{\sigma} \in \widetilde{\text{Gl}(V, V_+)}$ are elements such that $\pi(\widetilde{\tau}) = \tau$ and $\pi(\widetilde{\sigma}) = \sigma$, then one has a commutator pairing:

$$\{\tau, \sigma\}_{V_+}^V = \widetilde{\tau} \cdot \widetilde{\sigma} \cdot \widetilde{\tau}^{-1} \cdot \widetilde{\sigma}^{-1} \in k^\times.$$

Let us set elements $\sigma, \sigma', \tau, \tau' \in \text{Gl}(V, V_+)$ such that the σ 's commute with the τ 's. (But we need assume neither that $\sigma\sigma' = \sigma'\sigma$ nor that $\tau\tau' = \tau'\tau$.) Hence, according to the statements of [1], [2] and [6], the following relations hold:

- (1) $\{\sigma, \sigma\}_{V_+}^V = 1$.
- (2) $\{\sigma, \tau\}_{V_+}^V = \left(\{\tau, \sigma\}_{V_+}^V\right)^{-1}$.
- (3) $\{\sigma\sigma', \tau\}_{V_+}^V = \{\sigma, \tau\}_{V_+}^V \cdot \{\sigma', \tau\}_{V_+}^V$.
- (4) $\{\sigma, \tau\tau'\}_{V_+}^V = \{\sigma, \tau\}_{V_+}^V \cdot \{\sigma, \tau'\}_{V_+}^V$.
- (5) If $\sigma V_+ = V_+ = \tau V_+$, then we have:

$$(2.2) \quad \{\sigma, \tau\}_{V_+}^V = 1.$$
- (6) If $V_+ = \{0\}$ or $V_+ = V$, then $\{\sigma, \tau\}_{V_+}^V = 1$.
- (7) If $V_+ \subseteq \widetilde{V} \subseteq V$ and $\sigma\widetilde{V} = \widetilde{V} = \tau\widetilde{V}$, then

$$(2.3) \quad \{\sigma, \tau\}_{V_+}^V = \{\sigma, \tau\}_{V_+}^{\widetilde{V}}.$$
- (8) $\{\sigma, \tau\}_{V_+}^V$ depends only on the commensurability class of V_+ .
- (9) Let us assume that V is equipped with a direct sum decomposition $V = V^0 \oplus V^1$. Put $V_+^i := V^i \cap V_+$ for $i = 0, 1$ and let us assume that $V_+ = V_+^0 \oplus V_+^1$. Let commuting elements $\sigma_0, \sigma_1 \in \text{Gl}(V, V_+)$ be given such that

$$\sigma_i|_{V^0} \in \text{Gl}(V^0, V_+^0), \quad \sigma_i|_{V^1} = 1$$
 for $i = 0, 1$. We therefore have:

$$\{\sigma_0|_{V^0}, \sigma_1|_{V^0}\}_{V_+^0}^{V^0} = \{\sigma_0, \sigma_1\}_{V_+}^V.$$
- (10) Again, let us assume that V is equipped with a direct sum decomposition $V = V^0 \oplus V^1$, put $V_+^i := V^i \cap V_+$ for $i = 0, 1$ and let us assume that $V_+ = V_+^0 \oplus V_+^1$. Let $\sigma_0, \sigma_1 \in \text{Gl}(V, V_+)$ be given such that

$$\sigma_i|_{V^i} \in \text{Gl}(V^i, V_+^i), \quad \sigma_i|_{V^{1-i}} = 1$$
 for $i = 0, 1$. (Necessarily, σ_0 and σ_1 commute.) We then have:

$$\{\sigma_0, \sigma_1\}_{V_+}^V = (-1)^{\alpha_0\alpha_1},$$

where

$$\alpha_i := i(\sigma_i|_{V^i}, V_+^i, V^i) = i(\sigma_i, V_+, V)$$

for $i = 0, 1$.

- (11) If $V = V^0 \oplus V^1$ and $V_+ = V_+^0 \oplus V_+^1$, such that V^0 and V^1 are invariant by the action of two commuting elements $\tau, \sigma \in \text{Gl}(V, V_+)$ and $\tau|_{V^i}, \sigma|_{V^i} \in \text{Gl}(V^i, V_+^i)$, then:

$$(2.4) \quad \{\tau, \sigma\}_{V_+}^V = (-1)^\alpha \cdot \{\tau|_{V^0}, \sigma|_{V^0}\}_{V_+^0}^{V_+^0} \cdot \{\tau|_{V^1}, \sigma|_{V^1}\}_{V_+^1}^{V_+^1},$$

with

$$\begin{aligned} \alpha &= i(\tau|_{V^0}, V_+^0, V^0) \cdot i(\sigma|_{V^1}, V_+^1, V^1) \\ &\quad + i(\tau|_{V^1}, V_+^1, V^1) \cdot i(\sigma|_{V^0}, V_+^0, V^0). \end{aligned}$$

- (12) Let $V_- \subset V$ be a k -vector subspace of V such that $V = V_+ \oplus V_-$. If $\sigma, \tau \in \text{Gl}(V, V_+)$ are commuting elements, then $\sigma, \tau \in \text{Gl}(V, V_-)$ and

$$(2.5) \quad \{\sigma, \tau\}_{V_+}^V \cdot \{\sigma, \tau\}_{V_-}^V = 1.$$

- (13) If M and N are two k -vector subspaces of V , for all commuting elements $\tau, \sigma \in \text{Gl}(V, M) \cap \text{Gl}(V, N)$ one has that $\tau, \sigma \in \text{Gl}(V, M + N) \cap \text{Gl}(V, M \cap N)$ and:

$$(2.6) \quad \{\tau, \sigma\}_M^V \cdot \{\tau, \sigma\}_N^V = (-1)^\beta \cdot \{\tau, \sigma\}_{M+N}^V \cdot \{\tau, \sigma\}_{M \cap N}^V,$$

where

$$\begin{aligned} \beta &= i(\tau, M, V) \cdot i(\sigma, N, V) + i(\tau, N, V) \cdot i(\sigma, M, V) \\ &\quad + i(\tau, M + N, V) \cdot i(\sigma, M \cap N, V) + i(\tau, M \cap N, V) \cdot i(\sigma, M + N, V). \end{aligned}$$

Remark 2.2. Let C be a non-singular and irreducible curve over a perfect field k , and let Σ_C be its function field. If $x \in C$ is a closed point and we set $A_x = \widehat{\mathcal{O}_x}$ (the completion of the local ring \mathcal{O}_x), and $K_x = (\widehat{\mathcal{O}_x})_0$ (the field of fractions of $\widehat{\mathcal{O}_x}$, which coincides with the completion of Σ_C with respect to the valuation ring \mathcal{O}_x), it follows from [6] (Section 5) that, similar to [2], we have a central extension of groups

$$1 \rightarrow k^\times \rightarrow \widetilde{\text{Gl}}(K_x, A_x) \rightarrow \text{Gl}(K_x, A_x) \rightarrow 1,$$

which, since $\Sigma_C^* \subseteq \text{Gl}(K_x, A_x)$, induces by restriction another determinantal central extension of groups:

$$1 \rightarrow k^\times \rightarrow \widetilde{\Sigma}_C^\times \rightarrow \Sigma_C^\times \rightarrow 1,$$

whose commutator, for all $f, g \in \Sigma_C^\times$, is:

$$\{f, g\}_{A_x}^{K_x} = N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right] \in k^\times,$$

where $k(x)$ is the residue class field of the closed point x and $N_{k(x)/k}$ is the norm of the extension $k \hookrightarrow k(x)$.

Hence, according to [6] the tame symbol associated with a closed point $x \in C$ is the map

$$(\cdot, \cdot)_x : \Sigma_C^\times \times \Sigma_C^\times \longrightarrow k^\times ,$$

defined by:

$$(f, g)_x = (-1)^{\deg(x) \cdot v_x(f) \cdot v_x(g)} \cdot N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(p) \right]$$

for all $f, g \in \Sigma_C^\times$.

When x is a rational point of C , this definition coincides with the tame symbol associated with the field Σ_C and the discrete valuation v_x (the multiplicative local symbol -[8]-). Moreover, with the same method as Tate's proof of the Residue Theorem [9], using the properties of the commutator $\{\cdot, \cdot\}_{A_x}^{K_x}$ and the property of complete curves

$$\sum_{x \in C} \deg(x) v_x(f) = 0$$

for every $f \in \Sigma_C^\times$, from the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$ we have the reciprocity law:

$$\prod_{x \in C} (f, g)_x = 1$$

for all $f, g \in \Sigma_C^\times$.

2.2. New results on commutators. To conclude this section, we shall offer new results related to the commutator $\{\cdot, \cdot\}_{V_+}^V$ that will be used to develop the theory of reciprocity laws on arbitrary vector spaces.

First, we shall study the index of a direct sum of subspaces of V .

Lemma 2.3. *Assuming that $V_+ = H \oplus W$ and $\sigma \in \text{Gl}(V, H) \cap \text{Gl}(V, W)$, one has that*

$$i(\sigma, V_+, V) = i(\sigma, H, V) + i(\sigma, W, V) .$$

Proof. It is clear that $\sigma \in \text{Gl}(V, V_+)$ and, bearing in mind that

$$\begin{aligned} V_+ / (H \cap \sigma H + W \cap \sigma W) &\simeq H / (H \cap \sigma H) \oplus W / (W \cap \sigma W) \\ \sigma V_+ / (H \cap \sigma H + W \cap \sigma W) &\simeq \sigma H / (H \cap \sigma H) \oplus \sigma W / (W \cap \sigma W) , \end{aligned}$$

the claim can be deduced from the exact sequences:

$$\begin{aligned} 0 \rightarrow (V_+ \cap \sigma V_+) / (H \cap \sigma H + W \cap \sigma W) &\rightarrow V_+ / (H \cap \sigma H + W \cap \sigma W) \rightarrow V_+ / (V_+ \cap \sigma V_+) \rightarrow 0 \\ 0 \rightarrow (V_+ \cap \sigma V_+) / (H \cap \sigma H + W \cap \sigma W) &\rightarrow \sigma V_+ / (H \cap \sigma H + W \cap \sigma W) \rightarrow \sigma V_+ / (V_+ \cap \sigma V_+) \rightarrow 0 \end{aligned}$$

□

Lemma 2.4. *If $V_+ \sim H + W$ and $\sigma \in \text{Gl}(V, H) \cap \text{Gl}(V, W)$, then*

$$i(\sigma, V_+, V) = i(\sigma, H, V) + i(\sigma, W, V) - i(\sigma, H \cap W, V) .$$

Proof. Setting $H = (H \cap W) \oplus H'$ and $W = (H \cap W) \oplus W'$, since $H + W = (H \cap W) \oplus H' \oplus W'$ and $i(\sigma, H + W, V) = i(\sigma, V_+, V)$, the statement follows directly from Lemma 2.3. □

Corollary 2.5. *If $V_+ \sim H + W$ with $H \cap W \sim \{0\}$, and $\sigma \in \text{Gl}(V, H) \cap \text{Gl}(V, W)$, then*

$$i(\sigma, V_+, V) = i(\sigma, H, V) + i(\sigma, W, V) .$$

And, more generally, it is easy to check that:

Corollary 2.6. *If $V_+ \sim H_1 + \dots + H_n$ with $H_i \cap \sum_{j \neq i} H_j \sim \{0\}$, given $\sigma \in \bigcap_i \text{Gl}(V, H_i)$, then*

$$i(\sigma, V_+, V) = i(\sigma, H_1, V) + \dots + i(\sigma, H_n, V).$$

Moreover, it follows immediately from expressions (2.4) and (2.6) that:

Lemma 2.7. *With the above notation, if H, W are two arbitrary k -subspaces of V and $\sigma, \tau \in \text{Gl}(V, H) \cap \text{Gl}(V, W)$, then*

$$\{\tau, \sigma\}_{H \oplus W}^{V \oplus V} = \{\tau, \sigma\}_{[H+W] \oplus [H \cap W]}^{V \oplus V}.$$

Proposition 2.8. *If V_1, \dots, V_r are arbitrary k -subspaces of V , and we consider $\sigma, \tau \in \text{Gl}(V, V_1) \cap \dots \cap \text{Gl}(V, V_r)$, then:*

$$\{\tau, \sigma\}_{V_1 \oplus \dots \oplus V_r}^{V \oplus \dots \oplus V} = \{\tau, \sigma\}_{V_1 + \dots + V_r \oplus (V_1 + \dots + V_{r-1}) \cap V_r \oplus \dots \oplus (V_1 + V_2) \cap V_3 \oplus V_1 \cap V_2}^{V \oplus \dots \oplus V}.$$

Proof. Arguing by induction on r , the case $r = 2$ coincides with the statement of Lemma 2.7.

Hence, assuming that the statement holds for $r - 1$, and writing

$$\begin{aligned} \alpha_r &= i(\sigma, V_1 + \dots + V_{r-1} \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2, V^{r-1}) \cdot i(\tau, V_r, V) \\ &\quad + i(\tau, V_1 + \dots + V_{r-1} \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2, V^{r-1}) \cdot i(\sigma, V_r, V) \end{aligned}$$

and

$$\begin{aligned} \beta_r &= i(\sigma, V_1 + \dots + V_r \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2, V^{r-1}) \cdot i(\tau, (V_1 + \dots + V_{r-1}) \cap V_r, V) \\ &\quad + i(\tau, V_1 + \dots + V_r \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2, V^{r-1}) \cdot i(\sigma, (V_1 + \dots + V_{r-1}) \cap V_r, V), \end{aligned}$$

where $V^{r-1} := V \oplus \dots \oplus V$, it follows again from expressions (2.4) and (2.6) that:

$$\begin{aligned} &\{\tau, \sigma\}_{V_1 \oplus \dots \oplus V_r}^{V \oplus \dots \oplus V} = \\ &= (-1)^{\alpha_r} \cdot \{\tau, \sigma\}_{V_1 + \dots + V_{r-1} \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2}^{V \oplus \dots \oplus V} \cdot \{\tau, \sigma\}_{V_r}^V = \\ &= (-1)^{\alpha_r} \cdot \{\tau, \sigma\}_{V_1 + \dots + V_{r-1} \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2}^{V \oplus \dots \oplus V} \cdot \{\tau, \sigma\}_{V_r \oplus \{0\} \oplus \dots \oplus \{0\}}^{V \oplus \dots \oplus V} = \\ &= (-1)^{\beta_r} \cdot \{\tau, \sigma\}_{V_1 + \dots + V_r \oplus (V_1 + \dots + V_{r-2}) \cap V_{r-1} \oplus \dots \oplus V_1 \cap V_2}^{V \oplus \dots \oplus V} \\ &\quad \cdot \{\tau, \sigma\}_{(V_1 + \dots + V_{r-1}) \cap V_r \oplus \{0\} \oplus \dots \oplus \{0\}}^{V \oplus \dots \oplus V} = \\ &= \{\tau, \sigma\}_{V_1 + \dots + V_r \oplus (V_1 + \dots + V_{r-1}) \cap V_r \oplus \dots \oplus (V_1 + V_2) \cap V_3 \oplus V_1 \cap V_2}^{V \oplus \dots \oplus V}. \end{aligned}$$

□

Finally, also from expression (2.6) may be deduced that:

Lemma 2.9. *If $V = V_+ + \tilde{V}$ and $\tau, \sigma \in \text{Gl}(V, V_+)$ with $\sigma(\tilde{V}) = \tau(\tilde{V}) = \tilde{V}$, then $\tau, \sigma \in \text{Gl}(\tilde{V}, V_+ \cap \tilde{V})$ and*

$$\{\tau, \sigma\}_{V_+}^V = \{\tau, \sigma\}_{V_+ \cap \tilde{V}}^{\tilde{V}}.$$

3. RECIPROCITY LAW

3.1. General reciprocity law on vector spaces. Let us now consider an arbitrary infinite-dimensional vector k -space V .

Definition 3.1. We shall use the term “the restricted linear group” associated with an arbitrary family of k -subspaces $\{V_i\}_{i \in I}$ of V , $\text{Gl}(V, \{V_i\}_{i \in I})$, to refer to the subgroup of $\text{Aut}_k(V)$ defined by:

$$\begin{aligned} \text{Gl}(V, \{V_i\}_{i \in I}) = \{ \sigma \in \text{Aut}_k(V) \text{ such that } \sigma \in \text{Gl}(V, V_i) \text{ for all } i \\ \text{and } \sigma(V_i) = V_i \text{ for almost all } i \}. \end{aligned}$$

Let us now fix two families of subspaces of V , $\{V_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$, such that $V_i \subseteq W_i$ for all $i \in I$ and $\sigma(W_i) = W_i$ for every $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$.

Lemma 3.2. If $J = \{j_1, \dots, j_n\}$ is a finite subset of I , and we set $A_J = \bigoplus_{j \in J} V_j$ and $W_J = \bigoplus_{j \in J} W_j$, regarding $\text{Gl}(V, \{V_i\}_{i \in I})$ as a subgroup of $\text{Gl}(W_J, A_J)$ by means of the diagonal embedding $\sigma \mapsto (\sigma_j)$, where $\sigma_j = \sigma$ for all $j \in J$, then

$$i(\sigma, A_J, W_J) = \sum_{j \in J} i(\sigma, V_j, W_j)$$

for all $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$.

Proof. It is clear that to prove the claim we can assume that $\#J = 2$.

In this case, if $A_J = V_1 \oplus V_2$ and $W_J = W_1 \oplus W_2$, the statement follows directly from the isomorphisms of k -vector spaces:

$$\begin{aligned} V_1 \oplus V_2 / [V_1 \oplus V_2 \cap \sigma(V_1 \oplus V_2)] &\simeq V_1 / [V_1 \cap \sigma V_1] \oplus V_2 / [V_2 \cap \sigma V_2], \\ \sigma(V_1 \oplus V_2) / [V_1 \oplus V_2 \cap \sigma(V_1 \oplus V_2)] &\simeq \sigma(V_1) / [V_1 \cap \sigma V_1] \oplus \sigma(V_2) / [V_2 \cap \sigma V_2]. \end{aligned}$$

□

Lemma 3.3. If $J = \{j_1, \dots, j_n\}$ is a finite subset of I and we again set $A_J = \bigoplus_{j \in J} V_j$ and $W_J = \bigoplus_{j \in J} W_j$, one has that

$$\{\sigma, \tau\}_{A_J}^{W_J} = (-1)^{\sum_{\substack{j_s, j_r \in J \\ j_s \neq j_r}} i(\sigma, V_{j_s}, W_{j_s}) i(\tau, V_{j_r}, W_{j_r})} \cdot \prod_{j \in J} \{\sigma, \tau\}_{V_j}^{W_j}$$

for all commuting $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$.

Proof. Using induction over $\#J$, the formula holds from expression (2.4) and Lemma 3.2. □

Lemma 3.4. Let $J = \{j_1, \dots, j_n\}$ be a finite subset of I . If $A_J = \bigoplus_{j \in J} V_j$, $W_J = \bigoplus_{j \in J} W_j$, $A_I = \bigoplus_{i \in I} V_i$, $W_I = \bigoplus_{i \in I} W_i$, and $i(\sigma, A_I, W_I) = i(\tau, A_I, W_I) = 0$, then

$$\{\sigma, \tau\}_{A_J}^{W_J} = (-1)^{\sum_{j \in J} i(\sigma, V_j, W_j) i(\tau, V_j, W_j)} \cdot \prod_{j \in J} \{\sigma, \tau\}_{V_j}^{W_j},$$

for all commuting elements $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ such that $\sigma(V_i) = \tau(V_i) = V_i$ for all $i \in I - J$.

Proof. Bearing in mind that $i(\sigma, V_+, V) = 0$ when $\sigma \in \text{Gl}(V, V_+)$ and $\sigma(V_+) = V_+$, the statement is immediately deduced from Lemma 3.2 and Lemma 3.3. \square

Let us now write $A_I = \bigoplus_{i \in I} V_i$, $W_I = \bigoplus_{i \in I} W_i$. If $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$, it is clear that $\sigma, \tau \in \text{Gl}(W_I, A_I)$ by means of the diagonal embedding

Theorem 3.5 (General Reciprocity Law). *Let $\{V_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ be two families of subspaces of V such that $V_i \subseteq W_i$ for all $i \in I$ and $\sigma(W_i) = W_i$ for every $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$. For commuting elements $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ such that $\{\sigma, \tau\}_{A_I}^{W_I} = 1$ and $i(\sigma, A_I, W_I) = i(\tau, A_I, W_I) = 0$, then:*

$$\prod_{i \in I} (-1)^{i(\sigma, V_i, W_i) i(\tau, V_i, W_i)} \cdot \{\sigma, \tau\}_{V_i}^{W_i} = 1,$$

where only a finite number of terms are different from 1.

Proof. Let J be the finite subset of I containing only the points of I such that $\sigma(V_j) \neq V_j$ or $\tau(V_j) \neq V_j$. If we set $T = I - J$, we have that

$$W_I = W_T \oplus \bigoplus_{j \in J} W_j \quad \text{and} \quad A_I = A_T \oplus \bigoplus_{j \in J} V_j,$$

where $W_T = \bigoplus_{i \in T} W_i$ and $A_T = \bigoplus_{i \in T} V_i$.

Hence, since $\sigma, \tau \in \text{Aut}_k(A_T)$, it follows from expressions (2.2) and (2.4) that

$$\{\sigma, \tau\}_{A_J}^{W_J} = \{\sigma, \tau\}_{A_I}^{W_I} = 1.$$

Now, since J is finite and $\sigma(V_i) = \tau(V_i) = V_i$ for all $i \in T = I - J$, from Lemma 3.4 we have that

$$\{\sigma, \tau\}_{A_J}^{W_J} = (-1)^{\sum_{j \in J} i(\sigma, V_j, W_j) i(\tau, V_j, W_j)} \cdot \prod_{j \in J} \{\sigma, \tau\}_{V_j}^{W_j}.$$

And, finally, since $\{\sigma, \tau\}_{V_i}^{W_i} = 1$ and $i(\sigma, V_i, W_i) = i(\tau, V_i, W_i) = 0$ for all $i \in I - J$, then the statement is proved. \square

Let us now consider a commutative group G and a morphism of groups $\varphi: k^* \rightarrow G$. Hence, the central extension (2.1) induces an exact sequence of groups

$$1 \rightarrow G \rightarrow (\widetilde{\text{Gl}(V, V_+)})_\varphi \rightarrow \text{Gl}(V, V_+) \rightarrow 1$$

and, given commuting elements $\sigma, \tau \in \text{Gl}(V, V_+)$, we shall denote by $\{\sigma, \tau\}_{V_+, \varphi}^V$ the corresponding commutator. It is clear that

$$\{\sigma, \tau\}_{V_+, \varphi}^V = \varphi(\{\sigma, \tau\}_{V_+}^V) \in G.$$

Accordingly, a direct consequence of Theorem 3.5 is:

Corollary 3.6. *Let $\{V_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ be two families of subspaces of V such that $V_i \subseteq W_i$ for all $i \in I$ and $\sigma(W_i) = W_i$ for every $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$. For commuting elements $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ such that $\{\sigma, \tau\}_{A_I}^{W_I} = 1$ and $i(\sigma, A_I, W_I) = i(\tau, A_I, W_I) = 0$, if G is a commutative group and $\varphi: k^* \rightarrow G$ is a morphism of groups, then:*

$$\prod_{i \in I} [\varphi(-1)]^{i(\sigma, V_i, W_i) i(\tau, V_i, W_i)} \cdot \{\sigma, \tau\}_{V_i, \varphi}^{W_i} = 1,$$

where only a finite number of terms are different from 1.

3.2. Reciprocity-admissible families of subspaces. Let us consider again an arbitrary infinite-dimensional vector k -space V .

Definition 3.7. We shall use the term “reciprocity-admissible family” to refer to a set of subspaces $\{V_i\}_{i \in I}$ of V satisfying the following properties:

- $\sum_{i \in I} V_i \sim V$
- $V_i \cap [\sum_{j \neq i} V_j] \sim \{0\}$ for all $i \in I$.

Example 1. If $V = \bigoplus_{n \in \mathbb{N}} \langle e_n \rangle$, each map $\phi: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\#\{n \in \mathbb{N} \text{ with } n \notin \text{Im} \phi\} < \infty$$

$$\#\{(\alpha, \beta) \in \mathbb{N} \times \mathbb{N} \text{ s. t. there exists } (\gamma, \delta) \in \mathbb{N} \times \mathbb{N} \text{ with } \phi(\alpha, \beta) = \phi(\gamma, \delta)\} < \infty$$

determines the following reciprocity-admissible family of subspaces of V :

$$V_i = \bigoplus_{j \in \mathbb{N}} \langle e_{\phi(i, j)} \rangle.$$

In particular, every bijection $\varphi: \mathbb{N} \times \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ induces a reciprocity-admissible family of subspaces of V .

Moreover, similar to [3], we denote by $\text{Gl}(k)$ to the direct limit of the sequence

$$\text{Gl}(1, k) \subset \text{Gl}(2, k) \subset \text{Gl}(3, k) \subset \dots,$$

where each linear group $\text{Gl}(n, k)$ is injected into $\text{Gl}(n+1, k)$ by the correspondence

$$\tau \longmapsto \phi_n^{n+1}(\tau) = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $\text{Gl}(k)$ is a subgroup of the restricted linear group $\text{Gl}(V, \{V_i\}_{i \in I})$ associated with every reciprocity-admissible family $\{V_i\}_{i \in \mathbb{N}}$ defined above.

Henceforth in this subsection we shall consider a reciprocity-admissible family $\{V_i\}_{i \in I}$ of subspaces of V . We shall prove that $\{V_i\}_{i \in I}$ satisfies the conditions of Theorem 3.5, where $W_i = V$ for all $i \in I$. Therefore, we have a reciprocity law for commuting elements of the group $\text{Gl}(V, \{V_i\}_{i \in I})$.

Lemma 3.8. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , and $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$, then*

$$\sum_{i \in I} i(\sigma, V_i, V) = 0.$$

Proof. If S_σ is the subset of I such that $j \in S_\sigma \iff \sigma(V_j) = V_j$, and we write

$$V_{S_\sigma} = \sum_{j \in S_\sigma} V_j,$$

it is clear that $i(\sigma, V_{S_\sigma}, V) = 0$, and that $i(\sigma, V_j, V) = 0$ for all $j \in S_\sigma$.

Since $I = \{i_1, \dots, i_r\} \cup S_\sigma$, bearing in mind the definition of the reciprocity-admissible family of subspaces $\{V_i\}_{i \in I}$, it follows from Corollary 2.6 that:

$$\begin{aligned} 0 &= i(\sigma, V, V) = i(\sigma, V_{i_1}, V) + \dots + i(\sigma, V_{i_r}, V) + i(\sigma, V_{S_\sigma}, V) \\ &= i(\sigma, V_{i_1}, V) + \dots + i(\sigma, V_{i_r}, V) = \sum_{i \in I} i(\sigma, V_i, V). \end{aligned}$$

□

Corollary 3.9. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , $V_I = \bigoplus_{i \in I} V$ and $A_I = \bigoplus_{i \in I} V_i$, regarding $\text{Gl}(V, \{V_i\}_{i \in I})$ again as a subgroup of $\text{Gl}(V_I, A_I)$ by means of the diagonal embedding, then*

$$i(\sigma, A_I, W_I) = 0$$

for each $\sigma \in \text{Gl}(V, \{V_i\}_{i \in I})$.

Proposition 3.10. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , $J = \{j_1, \dots, j_n\}$ is a finite subset of I , $A_J = \bigoplus_{j \in J} V_j$ and $V_J = \bigoplus_{j \in J} V$, and $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ are commuting elements, then:*

$$\{\sigma, \tau\}_{A_J}^{V_J} = [\{\sigma, \tau\}_{\sum_{i \in I-J} V_i}^V]^{-1}.$$

Proof. Bearing in mind that $V_{i_1} \cap \dots \cap V_{i_r} \sim \{0\}$ for all $\{i_1, \dots, i_r\} \subset I$, it follows from expression (2.5) and Proposition 2.8 that

$$\{\sigma, \tau\}_{A_J}^{V_J} = \{\sigma, \tau\}_{\sum_{j \in J} V_j}^V = [\{\sigma, \tau\}_H^V]^{-1},$$

H being a k -subspace of V such that $[\sum_{j \in J} V_j] \oplus H = V$.

Thus, since $\sum_{i \in I} V_i \sim V$ and $V_i \cap (\sum_{j \neq i} V_j) \sim \{0\}$ for all $i \in I$, then $H \sim \sum_{i \in I-J} V_i$, and the claim is proved. \square

Proposition 3.11. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , for all commuting elements $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ one has that:*

$$\{\sigma, \tau\}_{A_I}^{V_I} = 1.$$

Proof. If J is the finite subset of I containing only the points of I such that $\sigma(V_j) \neq V_j$ or $\tau(V_j) \neq V_j$, and we set $T = I - J$, we have that

$$V_I = V_T \oplus \bigoplus_{j \in J} V \quad \text{and} \quad A_I = A_T \oplus \bigoplus_{j \in J} V_j.$$

Hence, since $\sigma, \tau \in \text{Aut}_k(A_T)$, it follows from expressions (2.2) and (2.4) that

$$\{\sigma, \tau\}_{A_I}^{V_I} = \{\sigma, \tau\}_{A_J}^{V_J}.$$

Moreover, since $\sum_{i \in I-J} V_i$ is invariant under the action of σ and τ , Proposition 3.10 shows that

$$\{\sigma, \tau\}_{A_J}^{V_J} = [\{\sigma, \tau\}_{\sum_{i \in I-J} V_i}^V]^{-1} = 1.$$

\square

Theorem 3.12. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , for all commuting elements $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ one has that:*

$$\prod_{i \in I} (-1)^{i(\sigma, V_i, V) i(\tau, V_i, V)} \cdot \{\sigma, \tau\}_{V_i}^V = 1,$$

where only a finite number of terms are different from 1.

Proof. Bearing in mind Corollary 3.9 and Proposition 3.11, the statement is a particular case of Theorem 3.5. \square

Similar to Corollary 3.6, a direct consequence of Theorem 3.12 is:

Corollary 3.13. *If $\{V_i\}_{i \in I}$ is a reciprocity-admissible family of subspaces of V , $\sigma, \tau \in \text{Gl}(V, \{V_i\}_{i \in I})$ such that $\sigma \circ \tau = \tau \circ \sigma$, G is a commutative group, and $\varphi: k^* \rightarrow G$ is a morphism of groups, then:*

$$\prod_{i \in I} [\varphi(-1)]^{i(\sigma, V_i, V) i(\tau, V_i, V)} \cdot \{\sigma, \tau\}_{V_i, \varphi}^V = 1,$$

where only a finite number of terms are different from 1.

3.3. Explicit expressions. The aim of the last part of this work is to provide explicit expressions of reciprocity laws associated with a reciprocity-admissible family of subspaces on an infinite-dimensional vector space, and to show that classical expressions can be deduced from Theorem 3.5, Corollary 3.6, Theorem 3.12 or Corollary 3.13.

3.3.1. Computations of $\text{Gl}(k)$. Let $\{V_i\}_{i \in I}$ be a reciprocity-admissible family of subspaces of $V = \bigoplus_{n \in \mathbb{N}} \langle e_n \rangle$ as in Example 1, and let $\sigma, \tau \in \text{Gl}(n_0, k) \subseteq \text{Gl}(k)$ be two commuting elements.

Since $\tilde{V} = \bigoplus_{j > n_0} \langle e_j \rangle$ is invariant under the action of σ and τ , since $V_i + \tilde{V} \sim V$, and since $V_i \cap \tilde{V}$ is also invariant under the action of σ and τ , it follows from Lemma 2.9 that:

$$\{\tau, \sigma\}_{V_i}^V = \{\tau, \sigma\}_{V_i \cap \tilde{V}}^{\tilde{V}} = 1.$$

Bearing in mind that from Lemma 2.4 one has that:

$$i(\phi, V_i, V) = i(\phi, V, V) - i(\phi, \tilde{V}, V) + i(\phi, \tilde{V} \cap V_i, V) = 0$$

for all $\phi \in \text{Gl}(n_0, k)$, the expression of the reciprocity law is confirmed.

To familiarize readers with calculations of objects referred to in this work, we now offer the direct computations of commutators and indices in an easy particular case with $\sigma, \tau \in \text{Gl}(2, k)$. We refer to ([6] -Sections 3 and 4-) for technical details.

Let us again consider the k -vector space $V = \bigoplus_{i \in \mathbb{N}} \langle e_i \rangle$ and let us now consider the k -subspaces:

- $V_1 = \bigoplus_{i \text{ odd}} \langle e_i \rangle.$
- $V_2 = \bigoplus_{j \text{ even}} \langle e_j \rangle.$

We shall now compute $\{\sigma, \tau\}_{V_1}^V$ and $\{\sigma, \tau\}_{V_2}^V$ where:

$$\sigma = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \tau = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix} \in \text{Gl}(2, k) \subset \text{Gl}(k) \text{ with } a, b, \lambda, \mu \in k^* \text{ and } \lambda b = \mu a.$$

Note that $V^* = \bigoplus_{i \in \mathbb{N}} \langle \omega_i \rangle$ with $\omega_i(e_j) = \delta_{ij}$.

Accordingly, for V_1 one has that:

- $V_1/[V_1 \cap \sigma V_1] = V_1/[V_1 \cap \tau V_1] \simeq \langle e_1 \rangle.$
- $(\sigma V_1/[V_1 \cap \sigma V_1])^* = (\tau V_1/[V_1 \cap \tau V_1])^* \simeq \langle \omega_2 \rangle.$

Hence, since

$$(\sigma^{-1})^* = \begin{pmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{pmatrix} \text{ and } (\tau^{-1})^* = \begin{pmatrix} 0 & \mu^{-1} \\ \lambda^{-1} & 0 \end{pmatrix},$$

if $\tilde{\sigma} = (\sigma, e_1 \otimes \omega_2), \tilde{\tau} = (\tau, e_1 \otimes \omega_2) \in \widetilde{\mathrm{Gl}(V, V_1)}$, bearing in mind that the law of group of $\mathrm{Gl}(V, V_1)$, then

$$\begin{aligned}\tilde{\sigma} \cdot \tilde{\tau} &= (\sigma \cdot \tau, (e_1 \otimes \omega_2) \cdot \bar{\sigma}(e_1 \otimes \omega_2)) \\ &= (\sigma \cdot \tau, (e_1 \otimes \omega_2) \cdot (ae_2 \otimes b^{-1}\omega_1)) = (\sigma \cdot \tau, ab^{-1}),\end{aligned}$$

and

$$\begin{aligned}\tilde{\tau} \cdot \tilde{\sigma} &= (\tau \cdot \sigma, (e_1 \otimes \omega_2) \cdot \bar{\tau}(e_1 \otimes \omega_2)) \\ &= (\tau \cdot \sigma, (e_1 \otimes \omega_2) \cdot (\lambda e_2 \otimes \mu^{-1}\omega_1)) = (\tau \cdot \sigma, \lambda\mu^{-1}).\end{aligned}$$

Therefore,

$$\{\sigma, \tau\}_{V_1}^V = \tilde{\sigma} \cdot \tilde{\tau} \cdot \tilde{\sigma}^{-1} \cdot \tilde{\tau}^{-1} = \tilde{\sigma} \cdot \tilde{\tau} \cdot (\tilde{\tau} \cdot \tilde{\sigma})^{-1} = ab^{-1}\mu\lambda^{-1} = 1.$$

Analogously, for V_2 we have that:

- $V_2/[V_2 \cap \sigma V_2] = V_2/[V_2 \cap \tau V_2] \simeq \langle e_2 \rangle.$
- $(\sigma V_2/[V_2 \cap \sigma V_2])^* = (\tau V_2/[V_2 \cap \tau V_2])^* \simeq \langle \omega_1 \rangle.$

Thus, if $\tilde{\sigma} = (\sigma, e_2 \otimes \omega_1), \tilde{\tau} = (\tau, e_2 \otimes \omega_1) \in \widetilde{\mathrm{Gl}(V, V_2)}$, similar to above it is possible to check that $\tilde{\sigma} \cdot \tilde{\tau} = (\sigma \cdot \tau, ba^{-1})$ and $\tilde{\tau} \cdot \tilde{\sigma} = (\tau \cdot \sigma, \mu\lambda^{-1})$. Then,

$$\{\sigma, \tau\}_{V_1}^V = ba^{-1}\lambda\mu^{-1} = 1,$$

and, in this case, both commutators are equal to 1.

And, since

$$i(\sigma, V_1, V) = i(\sigma, V_2, V) = i(\tau, V_1, V) = i(\tau, V_2, V) = 0,$$

then the reciprocity expression holds, and the triviality of the commutators and indices for commuting elements of $\mathrm{Gl}(k)$ is confirmed.

3.3.2. Weil Reciprocity Law. Similar to Remark 2.2, let C be a non-singular and irreducible curve over a perfect field k , and let Σ_C be its function field. If $x \in C$ is a closed point, we set $A_x = \widehat{\mathcal{O}_x}$ (the completion of the local ring \mathcal{O}_x), and $K_x = (\widehat{\mathcal{O}_x})_0$ (the field of fractions of $\widehat{\mathcal{O}_x}$, which coincides with the completion of Σ_C with respect to the valuation ring \mathcal{O}_x).

If we set

$$V = \prod'_{x \in C} K_x = \{f = (f_x) \text{ such that } f_x \in K_x \text{ and } f_x \in A_x \text{ for almost all } x\},$$

then Σ_C is regarded as a subspace of V by means of the diagonal embedding.

Let us now consider the family of k -subspaces of V consists of $V_1 = \bigoplus_{x \in C} A_x$ and $V_2 = \Sigma_C$. If C is a complete curve, since

$$V/(V_1 + V_2) \simeq H^1(C, \mathcal{O}_C) \text{ and } V_1 \cap V_2 = H^0(C, \mathcal{O}_C),$$

then $\{V_1, V_2\}$ is a reciprocity-admissible family of V .

Thus, bearing in mind that $\Sigma_C^\times \subset \mathrm{Gl}(V, \{V_1, V_2\})$ and that V_2 is invariant under the action of Σ_C^\times , Theorem 3.12 shows that

$$\{f, g\}_{V_1}^V = 1,$$

for all $f, g \in \Sigma_C^\times$.

Moreover, since $V = V_1 + \bigoplus_{x \in C} K_x$, $\bigoplus_{x \in C} K_x$ is invariant under the action of Σ_C^\times , and $V_1 \cap \bigoplus_{x \in C} K_x = \bigoplus_{x \in C} A_x$, then it follows from Lemma 2.9 that

$$\{f, g\}_{V_1}^V = \{f, g\}_{\bigoplus_{x \in C} A_x}^{\bigoplus_{x \in C} K_x} = 1,$$

for all $f, g \in \Sigma_C^\times$.

Hence, writing:

- $V_x = \cdots \oplus \{0\} \oplus A_x \oplus \{0\} \oplus \cdots$,
- $W_x = \cdots \oplus \{0\} \oplus K_x \oplus \{0\} \oplus \cdots$,

the families $\{V_x\}_{x \in C}$ and $\{W_x\}_{x \in C}$ of k -subspaces of V satisfy the hypothesis of Theorem 3.5 and, since $i(f, A_x, K_x) = \deg(x) \cdot v_x(f)$ and

$$\{f, g\}_{A_x}^{K_x} = N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right]$$

for all $f, g \in \Sigma_C^\times$, where $k(x)$ is the residue class field of the closed point x , $\deg(x) = \dim_k(k(x))$ and $N_{k(x)/k}$ is the norm of the extension $k \hookrightarrow k(x)$, we have that

$$\prod_{x \in C} (-1)^{\deg(x) v_x(f) v_x(g)} \cdot N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right] = 1 \in k^\times,$$

which is the explicit expression of the Weil Reciprocity Law.

Furthermore, with the same method one can analogously see that the reciprocity laws for the generalizations of the tame symbol to $\mathrm{Gl}(n, \Sigma_C)$ and $\mathrm{Gl}(\Sigma_C)$, offered by the author in [5], where C is again a complete, irreducible and non-singular curve over a perfect field, can also be deduced from Theorem 3.5 and Theorem 3.12.

Remark 3.14. Note that for this proof of the Weil Reciprocity Law it is not necessary to use the property of complete curves

$$\sum_{x \in C} \deg(x) v_x(f) = 0$$

for an arbitrary $f \in \Sigma_C^\times$. Hence, similar to Tate's proof of the Residue Theorem, from the above arguments we deduce that the Weil Reciprocity Law is only a direct consequence of the finiteness of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$ for complete curves.

3.3.3. Hilbert Norm Residue Symbol. Let C now be an irreducible, complete and non-singular curve over a finite perfect field k that contains the m^{th} roots of unity.

If $\#k = q$, one has the morphism of groups

$$\begin{aligned} \phi_m : k^\times &\longrightarrow \mu_m \\ x &\longmapsto x^{\frac{q-1}{m}}, \end{aligned}$$

and hence

$$\{f, g\}_{A_x, \phi_m}^{K_x} = N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right]^{\frac{q-1}{m}} \in \mu_m.$$

Arguing as above, from Corollary 3.6 and Corollary 3.13 we can deduce that

$$\begin{aligned} \prod_{x \in C} [\phi_m(-1)]^{i(f, A_x, K_x) \cdot i(g, A_x, K_x)} \cdot \{f, g\}_{A_x, \phi_m}^{K_x} &= \\ \prod_{x \in C} (-1)^{\deg(x) v_x(f) v_x(g) \frac{q-1}{m}} N_{k(x)/k} \left[\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right]^{\frac{q-1}{m}} &= \\ \prod_{x \in C} N_{k(x)/k} \left[(-1)^{v_x(f) \cdot v_x(g)} \cdot \frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right]^{\frac{q-1}{m}} &= 1, \end{aligned}$$

which is the explicit formula of the Reciprocity Law for the Hilbert Norm Residue Symbol given by H. L. Schmidt in [7]. And the same occurs with the reciprocity laws for the generalizations of the Hilbert Norm Residue Symbol to $\text{Gl}(n, \Sigma_C)$ and $\text{Gl}(\Sigma_C)$, also offered by the author in [5].

3.3.4. Reciprocity Law for Characters. Let V be an infinite-dimensional \mathbb{C} -vector space, and let $\{V_i\}_{i \in I}$ be an arbitrary reciprocity-admissible family of subspaces of V .

Let K be an abelian extension of \mathbb{C} such that K^\times is a subgroup of $\text{Gl}(V, \{V_i\}_{i \in I})$.

If μ_n is the group of the n^{th} roots of unity and $\mu_n = \langle \alpha_n \rangle$, for each V_i we have a map

$$\chi_{V_i, \alpha_n} : K^\times \longrightarrow \mathbb{C}^\times,$$

defined by $\chi_{V_i, \alpha_n}(f) = \{\alpha_n, f\}_{V_i}^V$. It is clear that χ_{V_i, α_n} is a character of order n .

Thus, since $i(\alpha_n, V_i) = 0$ for all $i \in I$, as a direct consequence of Theorem 3.12 we have the following reciprocity law for the family of characters $\{\chi_{V_i, \alpha_n}\}_{i \in I}$:

$$\prod_{i \in I} \chi_{V_i, \alpha_n}(f) = 1 \text{ for all } f \in K^\times.$$

In particular, again fixing a primitive n -root of unity α_n , if m is a positive integer, since $\mathbb{Z}/(m\mathbb{Z}) \simeq \mu_m$ and since μ_m is a subgroup of K^\times , then by restriction we have characters of order n :

$$\chi_{V_i, \alpha_n} : \mathbb{Z}/(m\mathbb{Z}) \longrightarrow \mathbb{C}^\times.$$

Remark 3.15. Let X be a complete algebraic curve over \mathbb{C} and let p be a prime integer. For each closed point $x \in X$, and fixing a primitive p -root of unity α_p , with the notation of Subsection 3.3.2 we have a character

$$\chi_{A_x, \alpha_p} : \Sigma_X^\times \longrightarrow \mathbb{C}^\times,$$

defined by $\chi_{A_x, \alpha_p}(f) = \{\alpha_p, f\}_{A_x}^{K_x} = \alpha_p^{v_x(f)}$.

Hence, we have that:

$$\prod_{x \in X} \chi_{A_x, \alpha_p}(f) = \prod_{x \in X} \alpha_p^{v_x(f)} = \alpha_p^{\sum_{x \in X} v_x(f)} = 1,$$

from where we deduce that $\sum_{x \in X} v_x(f) \equiv 0 \pmod{p}$ for all prime p , which implies the well-known formula $\sum_{x \in X} v_x(f) = 0$ for every $f \in \Sigma_X^\times$ with X a complete curve (“the number of zeros counted with multiplicity of a meromorphic function coincides with its number of poles counted with multiplicity”).

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